5 Interaction by particle exchange

In the modern understanding of particle physics, the interactions between particles are mediated by the exchange of force carrying gauge bosons. The rigorous theoretical formalism for describing these interactions is Quantum Field Theory, which is beyond the scope of this book. Here the concepts are developed in the context of relativistic quantum mechanics. The main purpose of this short chapter is to describe how interactions arise from the exchange of virtual particles and to provide an introduction to Quantum Electrodynamics.

5.1 First- and second-order perturbation theory

In quantum mechanics, the transition rate Γ_{fi} between an initial state i and a final state f is given by Fermi's golden rule $\Gamma_{fi} = 2\pi|T_{fi}|^2\rho(E_f)$, where T_{fi} is the transition matrix element, given by the perturbation expansion

$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} + \cdots .$$

The first two terms in the perturbation series can be viewed as "scattering in a potential" and "scattering via an intermediate state j" as indicated in Figure 5.1. In the classical picture of interactions, particles act as sources of fields that give rise to a potential in which other particles scatter.

In quantum mechanics, the process of scattering in a static potential corresponds to the first-order term in the perturbation expansion, $\langle f|V|i\rangle$. This picture of

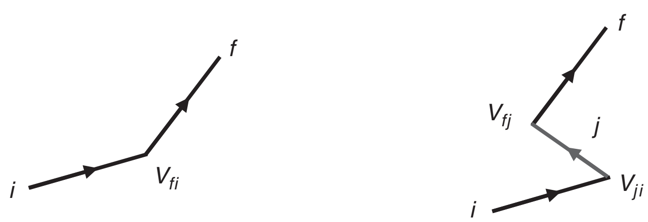


Fig. 5.1 Scattering in an external potential V_{fi} and scattering via an intermediate state, j.

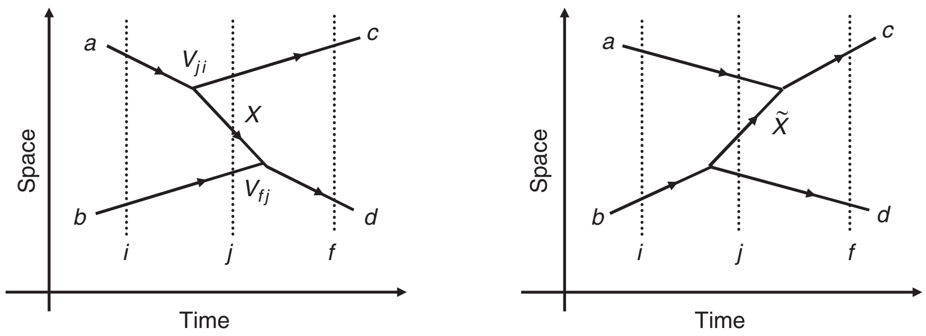


Fig. 5.2 Two possible time-orderings for the process $a + b \rightarrow c + d$.

scattering in the potential produced by another particle is unsatisfactory on a number of levels. When a particle scatters in a potential there is a transfer of momentum from one particle to another without any apparent mediating body. Furthermore, the description of forces in terms of potentials seems to imply that if a distant particle were moved suddenly, the potential due to that particle would change instantaneously at all points in space, seemingly in violation the special theory of relativity. In Quantum Field Theory, interactions between particles are mediated by the exchange of other particles and there is no mysterious action at a distance. The forces between particles result from the transfer of the momentum carried by the exchanged particle.

5.1.1 Time-ordered perturbation theory

The process of interaction by particle exchange can be formulated by using time-ordered perturbation theory. Consider the particle interaction, $a+b \rightarrow c+d$, which can occur via an intermediate state corresponding to the exchange of a particle X. There are two possible space-time pictures for this process, shown in Figure 5.2. In the first space-time picture, the initial state $|i\rangle$ corresponds to the particles $a+b$, the intermediate state $|j\rangle$ corresponds to $c + b + X$, and the final state $|f\rangle$ corresponds to $c + d$. In this time-ordered diagram, particle a can be thought of as emitting the exchanged particle X, and then at a later time X is absorbed by b. In QED this could correspond to an electron emitting a photon that is subsequently absorbed by a second electron. The corresponding term in the quantum-mechanical perturbation expansion is

$$T_{fi}^{ab} = \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} = \frac{\langle d|V|X + b\rangle\langle c + X|V|a\rangle}{(E_a + E_b) - (E_c + E_X + E_b)}. \tag{5.1}$$

The notation T_{fi}^{ab} refers to the time ordering where the interaction between a and X occurs before that between X and b. It should be noted that the energy of the intermediate state is not equal to that of the initial state, $E_j \neq E_i$, which is allowed for a short period of time by the energy–time uncertainty relation of quantum mechanics given by Equation (2.47). The interactions at the two vertices are defined by the non-invariant matrix elements $V_{ji} = \langle c + X|V|a\rangle$ and $V_{fj} = \langle d|V|X + b\rangle$. Following the arguments of Section 3.2.1, the non-invariant matrix element V_{ji} is related to the Lorentz-invariant (LI) matrix element $\mathcal{M}_{ji}$ by

$$V_{ji} = \mathcal{M}_{ji} \prod_k (2E_k)^{-1/2},$$

where the index k runs over the particles involved. In this case

$$V_{ji} = \langle c + X|V|a\rangle = \frac{\mathcal{M}_{a\to c+X}}{(2E_a 2E_c 2E_X)^{1/2}},$$

where $\mathcal{M}_{a\to c+X}$ is the LI matrix element for the fundamental interaction $a \to c + X$. The requirement that the matrix element $\mathcal{M}_{a\to c+X}$ is Lorentz invariant places strong constraints on its possible mathematical structure. To illustrate the concept of interaction by particle exchange, the simplest possible Lorentz-invariant coupling is assumed here, namely a scalar. In this case, the LI matrix element is simply $\mathcal{M}_{a\to c+X} = g_a$, and thus

$$V_{ji} = \langle c + X|V|a\rangle = \frac{g_a}{(2E_a 2E_c 2E_X)^{1/2}},$$

and the magnitude of the coupling constant g_a is a measure of the strength of the scalar interaction. Similarly

$$V_{fj} = \langle d|V|X + b\rangle = \frac{g_b}{(2E_b 2E_d 2E_X)^{1/2}},$$

where g_b is the coupling strength at the $b + X \to d$ interaction vertex. Therefore, with the assumed scalar form for the interaction, the second-order term in the perturbation series of (5.1) is

$$\begin{aligned} T_{fi}^{ab} &= \frac{\langle d|V|X + b\rangle\langle c + X|V|a\rangle}{(E_a + E_b) - (E_c + E_X + E_b)} \\ &= \frac{1}{2E_X} \cdot \frac{1}{(2E_a 2E_b 2E_c 2E_d)^{1/2}} \cdot \frac{g_a g_b}{(E_a - E_c - E_X)}. \end{aligned} \tag{5.2}$$

The LI matrix element for the process $a + b \to c + d$ is related to the corresponding transition matrix element by (3.9),

$$\mathcal{M}_{fi}^{ab} = (2E_a 2E_b 2E_c 2E_d)^{1/2} T_{fi}^{ab},$$

and thus from (5.2),

$$\mathcal{M}_{fi}^{ab} = \frac{1}{2E_X} \cdot \frac{g_a g_b}{(E_a - E_c - E_X)}. \tag{5.3}$$

The matrix element of (5.3) is Lorentz invariant in the sense that it is defined in terms of wavefunctions with an appropriate LI normalisation and has an LI scalar

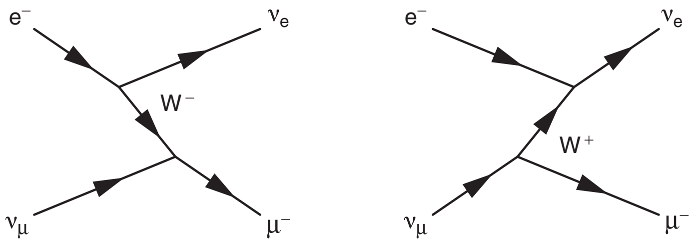


Fig. 5.3 The two lowest-order time-ordered diagrams contributing to $e^-\nu_\mu \to \nu_e\mu^-$ scattering.

form for the interaction. It should be noted that for this second-order process in perturbation theory, momentum is conserved at the interaction vertices but energy is not, $E_j \neq E_i$. Furthermore, the exchanged particle X satisfies the usual energy–momentum relationship, $E_X^2 = \mathbf{p}_X^2 + m_X^2$, and is termed "on-mass shell".

The second possible time-ordering for the process $a + b \to c + d$ is shown in the right-hand plot of Figure 5.2 and corresponds to b emitting $\tilde{X}$ which is subsequently absorbed by a. The exchanged particle $\tilde{X}$ in this time-ordering is assumed to have the same mass as X but has opposite charge(s). This must be the case if charge is to be conserved at each vertex. For example, in the process of $e^-\nu_\mu \to \nu_e\mu^-$ scattering, shown Figure 5.3, one of the time-ordered diagrams involves the exchange of a W^- and the other time-ordered diagram involves the exchange of a W^+. In the case of a QED process, there is no need to make this distinction for the neutral photon.

By repeating the steps that led to (5.3), it is straightforward to show that the LI matrix element for the second time-ordered diagram of Figure 5.2 is

$$\mathcal{M}_{fi}^{ba} = \frac{1}{2E_X} \cdot \frac{g_a g_b}{(E_b - E_d - E_X)}.$$

In quantum mechanics the different amplitudes for a process need to be summed to obtain the total amplitude. Here the total amplitude (at lowest order) is given by the sum of the two time-ordered amplitudes

$$\begin{aligned}\mathcal{M}_{fi} &= \mathcal{M}_{fi}^{ab} + \mathcal{M}_{fi}^{ba}\\ &= \frac{g_a g_b}{2E_X} \cdot \left(\frac{1}{E_a - E_c - E_X} + \frac{1}{E_b - E_d - E_X} \right),\end{aligned}$$

which, using energy conservation $E_b - E_d = E_c - E_a$, can be written

$$\begin{aligned}\mathcal{M}_{fi} &= \frac{g_a g_b}{2E_X} \cdot \left(\frac{1}{E_a - E_c - E_X} - \frac{1}{E_a - E_c + E_X} \right)\\ &= \frac{g_a g_b}{(E_a - E_c)^2 - E_X^2}.\end{aligned} \tag{5.4}$$

For both time-ordered diagrams, the energy of the exchanged particle E_X is related to its momentum by the usual Einstein energy–momentum relation, $E_X^2 = \mathbf{p}_X^2 + m_X^2$. Since momentum is conserved at each interaction vertex, for the first time-ordered process $\mathbf{p}_X = (\mathbf{p}_a - \mathbf{p}_c)$. In the case of the second time-ordered process $\mathbf{p}_{\tilde{X}} = (\mathbf{p}_b - \mathbf{p}_d) = -(\mathbf{p}_a - \mathbf{p}_c)$. Consequently, for *both* time-ordered diagrams the energy of the exchanged particle can be written as

$$E_X^2 = \mathbf{p}_X^2 + m_X^2 = (\mathbf{p}_a - \mathbf{p}_c)^2 + m_X^2.$$

Substituting this expression for E_X^2 into (5.4) leads to

$$\begin{aligned} \mathcal{M}_{fi} &= \frac{g_a g_b}{(E_a - E_c)^2 - (\mathbf{p}_a - \mathbf{p}_c)^2 - m_X^2} \\ &= \frac{g_a g_b}{(p_a - p_c)^2 - m_X^2}, \end{aligned} \tag{5.5}$$

where p_a and p_c are the respective four-momenta of particles a and c. Finally writing the four-momentum of the exchanged *virtual* particle X as

$$q = p_a - p_c,$$

gives

$$\mathcal{M}_{fi} = \frac{g_a g_b}{q^2 - m_X^2}. \tag{5.6}$$

This is a remarkable result. The sum over the two possible time-ordered diagrams in second-order perturbation theory has produced an expression for the interaction matrix element that depends on the four-vector scalar product q^2 and is therefore manifestly Lorentz invariant. In (5.6) the terms g_a and g_b are associated with the interaction vertices and the term

$$\boxed{\frac{1}{q^2 - m_X^2},} \tag{5.7}$$

is referred to as the propagator, is associated with the exchanged particle.

5.2 Feynman diagrams and virtual particles

In Quantum Field Theory, the sum over all possible time-orderings is represented by a *Feynman diagram*. The left-hand side of the diagram represents the initial state, and the right-hand side represents the final state. Everything in between represents the manner in which the interaction happened, regardless of the ordering in time. The Feynman diagram for the scattering process $a + b \to c + d$, shown in Figure 5.4, therefore represents the sum over the two possible time-orderings. The exchanged particles which appear in the intermediate state of a Feynman diagram, are referred to as *virtual particles*. A virtual particle is a mathematical construct representing the effect of summing over all possible time-ordered diagrams and,

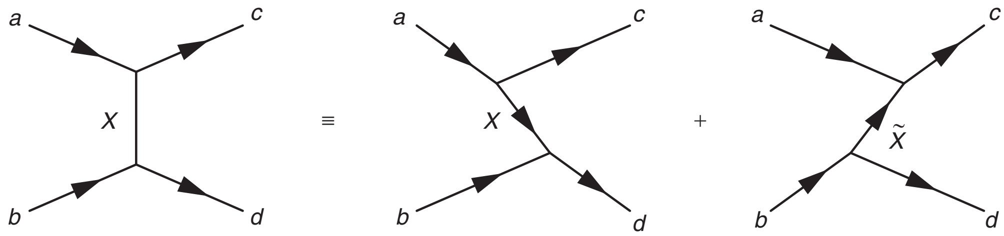


Fig. 5.4 The relation between the Feynman diagram for $a + b \to c + d$ scattering and the two possible time-ordered diagrams.

where appropriate, summing over the possible polarisation states of the exchanged particle.

From (5.5) it can be seen that the four-momentum q which appears in the propagator is given by the difference between the four-momenta of the particles entering and leaving the interaction vertex, $q = p_a - p_c = p_d - p_b$. Hence q can be thought of as the four-momentum of the exchanged virtual particle. By expressing the interaction in terms of the exchange of a virtual particle with four-momentum q, both momentum *and* energy are conserved at the interaction vertices of a Feynman diagram. This is not the case for the individual time-ordered diagrams, where energy is not conserved at a vertex. Because the q^2-dependence of the propagator is determined by the four-momenta of the incoming and outgoing particles, the virtual particle (which really represents the effect of the sum of all time-ordered diagrams) does not satisfy the Einstein energy–momentum relationship and it is termed off mass-shell, $q^2 \neq m_X^2$. Whilst the effects of the exchanged particles are observable through the forces they mediate, they are not directly detectable. To observe the exchanged particle would require its interaction with another particle and this would be a different Feynman diagram with additional (and possibly different) virtual particles.

The four-momentum q which appears in the propagator can be determined from the conservation of four-momentum at the interaction vertices. For example, Figure 5.5 shows the Feynman diagrams for the s-channel annihilation and the t-channel scattering processes introduced in Section 2.2.3. For the annihilation process, the four-momentum of the exchanged virtual particle is

$$q = p_1 + p_2 = p_3 + p_4,$$

and therefore $q^2 = (p_1 + p_2)^2$ which is the Mandelstam s variable. Previously (2.13) it was shown that $s = (E_1^* + E_2^*)^2$, where E_1^* and E_2^* are the energies of the initial-state particles in the centre-of-mass frame. Consequently, for an s-channel process $q^2 > 0$ and the exchanged virtual particle is termed "time-like" (the square of the time-like component of q is larger than the sum of the squares of the three space-like components). For the t-channel scattering diagram of Figure 5.5, the four momentum of the exchanged particle is given by $q = p_1 - p_3 = p_4 - p_2$. In this

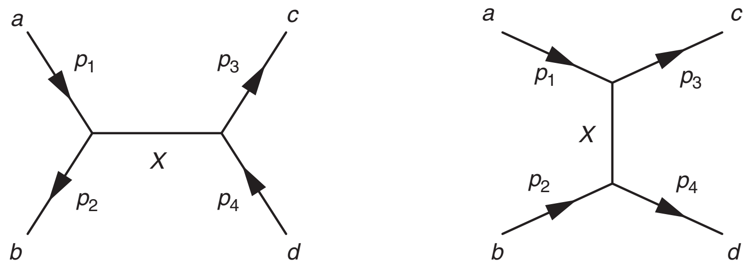


Fig. 5.5 Feynman diagrams for illustrative *s*-channel annihilation and *t*-channel scattering processes.

case q^2 is equal to the Mandelstam t variable. In Chapter 8 it will be shown that, for a t-channel process, q^2 is always less than zero and the exchanged virtual particle is termed "space-like".

5.2.1 Scattering in a potential

The covariant formulation of a scalar interaction in terms of the exchange of (virtual) particles leads to a Lorentz-invariant matrix element of the form

$$\mathcal{M}_{fi} = \frac{g_a g_b}{q^2 - m_X^2}\,. \tag{5.8}$$

This was derived by considering the second-order term in the perturbation expansion for T_{fi}. It is reasonable to ask how this picture of interaction by particle exchange relates to the familiar concept of scattering in a potential. For example, the differential cross section for the scattering of non-relativistic electrons ($\mathrm{v} \ll c$) in the electrostatic field of a stationary proton can be calculated using first perturbation theory with

$$M = \langle \psi_f | V(\mathrm{r}) | \psi_i \rangle = \int \psi_f^* V(\mathrm{r})\, \psi_i \, \mathrm{d}^3\mathbf{r}, \tag{5.9}$$

where $V(\mathrm{r})$ is the effective *static* electrostatic potential due to the proton and ψ_i and ψ_f are the wavefunctions of the initial and final-state electron. In the non-relativistic limit, this approach successfully reproduces the experimental data. However, the concept of scattering from a static potential is intrinsically not Lorentz invariant; the integral in matrix element of (5.9) only involves spatial coordinates.

The covariant picture of scattering via particle exchange applies equally in the non-relativistic and highly relativistic limits. In the non-relativistic limit, the form of the static potential used in first-order perturbation theory is that which reproduces the results of the more general treatment of the scattering process in terms of particle exchange. For example, the form of the potential $V(\mathrm{r})$ that reproduces the low-energy limit of scattering with the matrix element of (5.8) is the

Yukawa potential

$$V(\mathrm{r}) = g_a g_b \frac{e^{-m\mathrm{r}}}{\mathrm{r}}.$$

In this way, it is possible relate the formalism of interaction by particle exchange to the more familiar (non-relativistic) concept of scattering in a static potential. For an interaction involving the exchange of a massless particle, such as the photon, the Yukawa potential reduces to the usual $1/\mathrm{r}$ form of the Coulomb potential.

5.3 Introduction to QED

Quantum Electrodynamics (QED) is the Quantum Field Theory of the electromagnetic interaction. A first-principles derivation of the QED interaction from QFT goes beyond the scope of this book. Nevertheless, the basic interaction and corresponding Feynman rules can be obtained following the arguments presented in Section 5.1.1. The LI matrix element for a scalar interaction, given in (5.6), is composed of three parts: the strength of interaction at each of the two vertices, $\langle\psi_c|V|\psi_a\rangle$ and $\langle\psi_d|V|\psi_b\rangle$, and the propagator for the exchanged virtual particle of mass m_X, which can be written as

$$\mathcal{M} = \langle\psi_c|V|\psi_a\rangle \frac{1}{q^2 - m_X^2} \langle\psi_d|V|\psi_b\rangle. \tag{5.10}$$

In the previous example, the simplest Lorentz-invariant choice for the interaction vertex was used, namely a scalar interaction of the form $\langle\psi|V|\phi\rangle \propto g$. To obtain the QED matrix element for a scattering process, such as that shown in Figure 5.6, the corresponding expression for the QED interaction vertex is required. Furthermore, for the exchange of the photon, which is a spin-1 particle, it is necessary to sum over the quantum-mechanical amplitudes for the possible polarisation states.

The free photon field A_μ can be written in terms of a plane wave and a four-vector $\varepsilon^{(\lambda)}$ for the polarisation state λ,

$$A_\mu = \varepsilon_\mu^{(\lambda)} e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}.$$

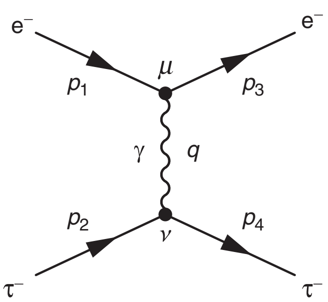


Fig. 5.6 The Feynman diagram for the QED scattering process $e^-\tau^- \to e^-\tau^-$.

The properties of the free photon field in classical electromagnetism are discussed in detail in Appendix D. For a real (as opposed to virtual) photon, the polarisation vector is always transverse to the direction of motion. Thus, a photon propagating in the z-direction can be described by two orthogonal polarisation states

$$\varepsilon^{(1)} = (0, 1, 0, 0) \quad \text{and} \quad \varepsilon^{(2)} = (0, 0, 1, 0).$$

The fundamental interaction between a fermion with charge q and an electromagnetic field described by a four-potential $A_\mu = (\phi, \mathbf{A})$ can be obtained by making the minimal substitution (see Section 4.7.5)

$$\partial_\mu \to \partial_\mu + iqA_\mu \,,$$

where $A_\mu = (\phi, -\mathbf{A})$ and $\partial_\mu = (\partial/\partial t, +\boldsymbol{\nabla})$. With this substitution, the free-particle Dirac equation becomes

$$\gamma^\mu \partial_\mu \psi + iq\gamma^\mu A_\mu \psi + im\psi = 0. \tag{5.11}$$

This is the wave equation for a spin-half particle in the presence of the electromagnetic field A_μ. The interaction Hamiltonian can be obtained by pre-multiplying (5.11) by $i\gamma^0$ to give

$$i\frac{\partial\psi}{\partial t} + i\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\psi - q\gamma^0\gamma^\mu A_\mu\psi - m\gamma^0\psi = 0,$$

where $\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}$ is shorthand for $\gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z}$. Since

$$\hat{H}\psi = i\frac{\partial\psi}{\partial t},$$

the Hamiltonian for a spin-half particle in an electromagnetic field can be identified as

$$\hat{H} = (m\gamma^0 - i\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}) + q\gamma^0\gamma^\mu A_\mu. \tag{5.12}$$

The first term on the RHS of (5.12) is just the free-particle Hamiltonian $\hat{H}_D$ already discussed in Chapter 4, and therefore can be identified as the combined rest mass and kinetic energy of the particle. The final term on the RHS of (5.12) is the contribution to the Hamiltonian from the interaction and thus the potential energy operator can be identified as

$$\hat{V}_D = q\gamma^0\gamma^\mu A_\mu. \tag{5.13}$$

This result appears reasonable since the time-like ($\mu = 0$) contribution to $\hat{V}_D$ is $q\gamma^0\gamma^0 A_0 = q\phi$, which is just the normal expression for the energy of a charge q in the scalar potential ϕ.

The Lorentz-invariant matrix element for the QED process of $e^-\tau^- \to e^-\tau^-$ scattering, shown in Figure 5.6, can be obtained by using the potential of (5.13) for the interaction at the $e^-\gamma$ vertex (labelled by the index μ)

$$\langle\psi(p_3)|\hat{V}_D|\psi(p_1)\rangle \to u_e^\dagger(p_3)\, Q_e e\gamma^0\gamma^\mu\varepsilon_\mu^{(\lambda)}\, u_e(p_1),$$

where the charge $q = Qe$ is expressed in terms of the magnitude of charge of the electron (such that $Q_e = -1$). Since the wavefunctions are four-component spinors, the final-state particle necessarily appears as the Hermitian conjugate $u^\dagger(p_3) \equiv u^{*T}(p_3)$ rather than $u^*(p_3)$. Similarly, the interaction at the $\tau^-\gamma$ vertex (labelled ν) can be written as

$$u_\tau^\dagger(p_4)\, Q_\tau e\gamma^0\gamma^\nu\varepsilon_\nu^{(\lambda)*}\, u_\tau(p_2).$$

The QED matrix element is obtained by summing over both the two possible time orderings and the possible polarisation states of the virtual photon. The sum over the two time-ordered diagrams follows directly from the previous result of (5.10). Hence the Lorentz-invariant matrix element for this QED process, which now includes the additional sum over the photon polarisation, is

$$\mathcal{M} = \sum_\lambda \left[u_e^\dagger(p_3)Q_e e\gamma^0\gamma^\mu u_e(p_1)\right]\varepsilon_\mu^{(\lambda)}\frac{1}{q^2}\varepsilon_\nu^{(\lambda)*}\left[u_\tau^\dagger(p_4)Q_\tau e\gamma^0\gamma^\nu u_\tau(p_2)\right]. \tag{5.14}$$

In Appendix D.4.3, it is shown that the sum over the polarisation states of the virtual photon can be taken to be

$$\sum_\lambda \varepsilon_\mu^{(\lambda)}\varepsilon_\nu^{(\lambda)*} = -g_{\mu\nu},$$

and therefore (5.14) becomes

$$\mathcal{M} = \left[Q_e e\, u_e^\dagger(p_3)\gamma^0\gamma^\mu u_e(p_1)\right]\frac{-g_{\mu\nu}}{q^2}\left[Q_\tau e\, u_\tau^\dagger(p_4)\gamma^0\gamma^\nu u_\tau(p_2)\right]. \tag{5.15}$$

This can be written in a more compact form using the adjoint spinors defined by $\overline{\psi} = \psi^\dagger\gamma^0$,

$$\mathcal{M} = -[Q_e e\, \overline{u}_e(p_3)\gamma^\mu u_e(p_1)]\frac{g_{\mu\nu}}{q^2}[Q_\tau e\, \overline{u}_\tau(p_4)\gamma^\nu u_\tau(p_2)]. \tag{5.16}$$

In Appendix B.3 it is shown that the combination of spinors and γ-matrices $j^\mu = \overline{u}(p)\gamma^\mu u(p')$ forms as contravariant four-vector under Lorentz boosts. By writing the four-vector currents

$$j_e^\mu = \overline{u}_e(p_3)\gamma^\mu u_e(p_1) \quad \text{and} \quad j_\tau^\nu = \overline{u}_\tau(p_4)\gamma^\nu u_\tau(p_2). \tag{5.17}$$

Equation (5.16) can be written in the manifestly Lorentz-invariant form of a four-vector scalar product

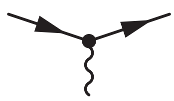
$$\mathcal{M} = -Q_e Q_\tau\, e^2\, \frac{j_e \cdot j_\tau}{q^2}\,. \tag{5.18}$$

This demonstrates that the interaction potential of (5.13) gives rise to a Lorentz-invariant description of the electromagnetic interaction.

5.4 Feynman rules for QED

A rigorous derivation of the matrix element of (5.16) can be obtained in the framework of quantum field theory. Nevertheless, the treatment described here shares some of the features of the full QED derivation, namely the summation over all possible time-orderings and polarisation states of the massless photon which gives rise to the photon propagator term $g_{\mu\nu}/q^2$, and the $Qe\bar{u}\gamma^\mu u$ form of the QED interaction between a fermion and photon. The expression for the matrix element of (5.16) hides a lot of complexity. If every time we were presented with a new Feynman diagram, it was necessary to derive the matrix element from first principles, this would be extremely time consuming. Fortunately this is not the case; the matrix element for *any* Feynman diagram can be written down immediately by following a simple set of rules that are derived formally from QFT.

There are three basic elements to the matrix element corresponding to the Feynman diagram of Figure 5.6: (i) the Dirac spinors for the external fermions (the initial- and final-state particles); (ii) a propagator term for the virtual photon; and (iii) a vertex factor at each interaction vertex. For each of these elements of the Feynman diagram, there is a *Feynman rule* for the corresponding term in the matrix element. The product of all of these terms is equivalent to $-i\mathcal{M}$. In their simplest form, the Feynman rules for QED, which can be used to calculate lowest-order cross sections, are as follows.

initial-state particle:	$u(p)$	
final-state particle:	$\bar{u}(p)$	
initial-state antiparticle:	$\bar{v}(p)$	
final-state antiparticle:	$v(p)$	
initial-state photon:	$\varepsilon_\mu(p)$	
final-state photon:	$\varepsilon^*_\mu(p)$	
photon propagator:	$-\dfrac{ig_{\mu\nu}}{q^2}$	
fermion propagator:	$-\dfrac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2}$	
QED vertex:	$-iQe\gamma^\mu$	

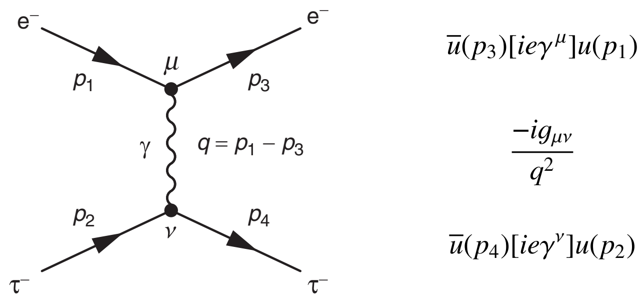


Fig. 5.7 The Feynman diagram for the QED scattering process $e^-\tau^- \to e^-\tau^-$ and the associated elements of the matrix element constructed from the Feynman rules. The matrix element is comprised of a term for the electron current, a term for the tau-lepton current and a term for the photon propagator.

It should be noted that in QED, the fundamental interaction is between a single photon and two spin-half fermions; there is no QED vertex connecting more than three particles. For this reason, all valid QED processes are described by Feynman diagrams formed from the basic three-particle QED vertex.

The use of the Feynman rules is best illustrated by example. Consider again the Feynman diagram for the process $e^-\tau^- \to e^-\tau^-$, shown in Figure 5.7. The indices μ and ν label the two interaction vertices. Applying the Feynman rules to the electron current, gives an adjoint spinor for the final-state electron, a factor $ie\gamma^\mu$ for the interaction vertex labelled by μ, and a spinor for the initial-state electron. The adjoint spinor is always written first and thus the contribution to the matrix element from the electron current is

$$\overline{u}(p_3)[ie\gamma^\mu]u(p_1).$$

The same procedure applied to the tau-lepton current gives

$$\overline{u}(p_4)[ie\gamma^\nu]u(p_2).$$

Finally, the photon propagator contributes a factor

$$\frac{-ig_{\mu\nu}}{q^2}.$$

The product of these three terms gives $-i\mathcal{M}$ and therefore

$$-i\mathcal{M} = [\overline{u}(p_3)\{ie\gamma^\mu\}u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_4)\{ie\gamma^\nu\}u(p_2)], \tag{5.19}$$

which is equivalent to the expression of (5.16), which was obtained from first principle arguments.

5.4.1 Treatment of antiparticles

The Feynman diagram for the s-channel annihilation process $e^+e^- \to \mu^+\mu^-$ is shown in Figure 5.8. Antiparticles are represented by lines in the negative time direction, reflecting the interpretation of the negative energy solutions to Dirac equation as particles which travel backwards in time. It is straightforward to obtain the matrix element for $e^+e^- \to \mu^+\mu^-$ from the Feynman rules. The part of the matrix element due to the electron and muon currents are, respectively,

$$\overline{v}(p_2)[ie\gamma^\mu]u(p_1) \quad \text{and} \quad \overline{u}(p_3)[ie\gamma^\nu]v(p_4),$$

where v-spinors are used to describe the antiparticles. As before, the photon propagator is

$$\frac{-ig_{\mu\nu}}{q^2}.$$

Hence the matrix element for $e^+e^- \to \mu^+\mu^-$ annihilation is given by

$$-i\mathcal{M} = [\overline{v}(p_2)\{ie\gamma^\mu\}u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_3)\{ie\gamma^\nu\}v(p_4)]. \tag{5.20}$$

The QED matrix element for the s-channel annihilation process $e^+e^- \to \mu^+\mu^-$ given by (5.20) is very similar to that for the t-channel scattering process $e^-\tau^- \to e^-\tau^-$ given by (5.19). Apart from the presence of the v-spinors for antiparticles, the only difference is the order in which the particles appear in the expressions for the currents. Fortunately, it is not necessary to remember the Feynman rules that specify whether a particle/antiparticle appears in the matrix element as a spinor or as an adjoint spinor, there is an easy mnemonic; the first particle encountered when following the line representing a fermion current from the end to the start in the direction *against* the sense of the arrows, always appears as the adjoint spinor. For example, in Figure 5.8, the incoming e^+ and outgoing μ^- are written as adjoint spinors.

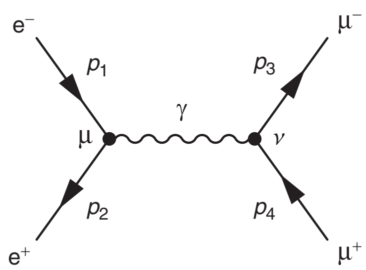


Fig. 5.8 The lowest-order Feynman diagram for the QED annihilation process $e^+e^- \to \mu^+\mu^-$.

Summary

This chapter described the basic ideas behind the description of particle interactions in terms of particle exchange and provided an introduction to the Feynman rules of QED. A number of important concepts were introduced. The sum of all possible time-ordered diagrams results in a Lorentz-invariant (LI) matrix element including propagator terms for the exchanged virtual particles of the form

$$\frac{1}{q^2 - m_X^2}.$$

The four-momentum appearing in the propagator term was shown to be determined by energy and momentum conservation at the interaction vertices.

The matrix element for a particular process is then constructed from propagator terms for the virtual particles and vertex factors. In QED, the interaction between a photon and a charged fermion has the form

$$ieQ\,\overline{u}_f\gamma^\mu u_i,$$

where u_i is the spinor for the initial-state particle and $\overline{u}_f$ is the adjoint spinor for the final-state particle. Finally, for each element of a Feynman diagram there is a corresponding Feynman rule which can be used to construct the matrix element for the diagram.

Problems

5.1 Draw the two time-ordered diagrams for the *s*-channel process shown in Figure 5.5. By repeating the steps of Section 5.1.1, show that the propagator has the same form as obtained for the *t*-channel process.

Hint: one of the time-ordered diagrams is non-intuitive, remember that in second-order perturbation theory the intermediate state does not conserve energy.

5.2 Draw the *two* lowest-order Feynman diagrams for the Compton scattering process $\gamma e^- \to \gamma e^-$.

5.3 Draw the lowest-order *t*-channel and *u*-channel Feynman diagrams for $e^+e^- \to \gamma\gamma$ and use the Feynman rules for QED to write down the corresponding matrix elements.